

The nineteen-vertex model and alternating sign matrices

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Abstract

It is shown that the transfer matrix of the inhomogeneous nineteen-vertex model with certain diagonal twisted boundary conditions possesses a simple eigenvalue. This is achieved through the identification of a simple and completely explicit solution of its Bethe equations. The corresponding eigenvector is computed by means of the algebraic Bethe ansatz, and both a simple component and its square norm are expressed in terms of the Izergin-Korepin determinant. In the homogeneous limit, the vector coincides with a supersymmetry singlet of the twisted spin-one XXZ chain. It is shown that in a natural polynomial normalisation scheme its square norm and the simple component coincide with generating functions for weighted enumeration of alternating sign matrices.

1 Introduction

The study of integrable quantum spin chains relies often on the solution of the Bethe equations [1, 2]. These are a set of non-linear coupled equations for the so-called Bethe roots, involving rational, trigonometric or even elliptic functions depending to the case at hand. Their solutions allow to build eigenstates of the Hamiltonian, and are thus in principle the starting point for the calculation of physically relevant quantities such as correlation functions. However, solving these equations is in general a challenging problem. Especially in finite-size systems one has to recourse frequently to numerical solutions because the patterns of Bethe roots for the ground state of the system and low-lying excited states are typically very complicated. Simplifications occur in infinite systems where the Bethe equations can be transformed into integral equations for which various analytical solution methods exist.

It is therefore to be expected that non-trivial integrable models for which the exact finite-size Bethe roots for the ground state can be computed (explicitly) by analytical methods are rather scarce. One of the few examples is the spin- $1/2$ XXZ chain with anisotropy $\Delta = -1/2$. Indeed, the Q -function, a polynomial whose roots coincide with the Bethe roots, was found exactly for the ground state of finite chains with and various boundary conditions: periodic, twisted and open [3, 4]. Extensions to the spin- $1/2$ XYZ along a particular line of couplings are known, and the corresponding Q -function displays even remarkable relations to classically integrable equations [5, 6]. It seems however that in each of these cases the boundary conditions have to be fine-tuned: twist angles or boundary

magnetic fields need to be adjusted to particular values, the system length has to be even or odd depending on the particular choice of boundary conditions, etc.

The purpose of this article is to present an example of a quantum integrable model for which the Bethe roots of a highly non-trivial eigenstate of the Hamiltonian can be exactly and explicitly determined in finite size. It is the spin-one XXZ chain [7, 8] with particular, fine-tuned twisted boundary conditions but *arbitrary anisotropy*. In [9], this spin chain was shown to possess a supersymmetric structure on the lattice in certain (anti-)cyclic subsectors of its Hilbert space. The ground states in these sectors are the so-called supersymmetry singlets. We show that at least one of these singlets can be obtained from the Bethe ansatz when all the Bethe roots coincide. This case is known to be difficult to handle as it has to be defined through a suitable limiting procedure [10] (at least within the framework of the coordinate Bethe ansatz). In order to circumvent this technical obstacle, we use the common trick to introduce inhomogeneities into the model in such a way that its integrability is preserved. While the notion of a local spin-chain Hamiltonian and lattice supersymmetry are absent in the inhomogeneous case, the transfer matrix of the corresponding nineteen-vertex model remains a well-defined object to study. We identify a special boundary condition with a fine-tuned twist angle for which it possesses a simple eigenvalue with a corresponding eigenstate whose Bethe roots are shown to coincide simply with the inhomogeneity parameters.

The existence of explicit Bethe roots and a simple eigenvalue does of course not necessarily imply that the corresponding eigenstate is interesting. Yet, a look at the literature on the above-mentioned spin- $1/2$ XXZ and XYZ chains, and the vertex- models which they are related to, shows that these states are in fact the ground states of the spin chains, and that in suitable normalisation their components display remarkable connections to problems of enumerative combinatorics, most importantly the enumeration of alternating sign matrices and plane partitions [11, 12, 13, 14, 15]. A fruitful approach to proving these properties is indeed the introduction of inhomogeneity parameters which allowed to analyse the eigenvectors in terms of the so-called quantum Knizhnik-Zamolodchikov system. Its polynomial solutions allow to determine various components, sum rules, and even exact finite-size correlation functions in the inhomogeneous case, and then take the homogeneous limit, see for example [16, 17, 18, 19]. We show here that a similar rich structure can be found in the transfer-matrix eigenstate of the twisted inhomogeneous nineteen-vertex model corresponding to the simple eigenvalue. Furthermore, we present relations to problems of weighted enumerations of alternating sign matrices in the homogeneous limit. Our approach is based on the explicit construction of the eigenstate by means of the algebraic Bethe ansatz, and the analysis of its properties through known results on scalar products such as Slavnov's formula [20, 21]. We show that with a suitable non-trivial normalisation convention a certain simple component and the square norm of the supersymmetry singlet coincide with generating functions for the weighted enumeration of alternating sign matrices.

The layout of this article is the following. We start in section 2 with a discussion of the quantum spin-one XXZ chain, recall briefly its lattice supersymmetry and state our results about its supersymmetry singlets. Section 3 is a reminder on the construction of the nineteen-vertex model from the fusion procedure. We prove the existence of a simple eigenvalue of the transfer matrix and thus of the spin-chain Hamiltonian in section 4. Starting from elementary properties of the corresponding eigenvector in section 5 we prove a relation between the square norm of the inhomogeneous eigenstate and the so-called Izergin-Korepin determinant, and use it to find a closed expression for a particular component of the vector. The evaluation of their homogeneous limit yields a relation between the supersymmetry singlet and alternating sign matrices. We present our conclusions in section 6.

2 The spin-one XXZ chain

The purpose of this section is to recall the definition of the spin-one XXZ chain with diagonal twisted boundary conditions as well as its supersymmetry properties, and then state our results about a particular supersymmetry singlet and its relation to the enumeration of alternating sign matrices.

Hilbert space and spin operators. The Hilbert space of the quantum spin chain with N sites is given by

$$V = V_1 \otimes V_2 \otimes \cdots \otimes V_N, \quad (1)$$

where every factor is a copy of the Hilbert space for a single spin one, $V_j \simeq \mathbb{C}^3$. We label the canonical basis vectors as follows

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The most simple choice of a basis of V is the set of vectors $|\sigma_1, \dots, \sigma_N\rangle = \bigotimes_{j=1}^N |\sigma_j\rangle$ where $\sigma_j = \uparrow, 0, \downarrow$ for all $j = 1, \dots, N$.

The spins themselves are described by the spin-one representation of $\mathfrak{su}(2)$ which is given by

$$s^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad s^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad s^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

As usual, we denote by s_j^a the operator s^a acting on the j -th factor of the tensor product (1). The total magnetisation is given by the operator $\Sigma = \sum_{j=1}^N s_j^3$. It is diagonal in the canonical basis. Moreover, since we are going to consider periodic systems, it will be useful to introduce a shift operator which translates the system by one site $S : V_1 \otimes V_2 \otimes \cdots \otimes V_N \mapsto V_N \otimes V_1 \otimes \cdots \otimes V_{N-1}$, and hence transforms the spins according to $S s_j^a S^{-1} = s_{j+1}^a$ for $j = 1, \dots, N-1$, and $S s_N^a S^{-1} = s_1^a$.

Hamiltonian. The Hamiltonian of the spin chain considered in this article is given by [7, 8]

$$H = \sum_{j=1}^N \left(\sum_{a=1}^3 J_a (s_j^a s_{j+1}^a + 2(s_j^a)^2) - \sum_{a,b=1}^3 A_{ab} s_j^a s_j^b s_{j+1}^a s_{j+1}^b \right), \quad (2a)$$

where A is a symmetric matrix $A_{ab} = A_{ba}$ with diagonal elements $A_{aa} = J_a$. The remaining constants depend only on a single parameter x which measures the anisotropy of the spin chain. They are given by

$$J_1 = J_2 = 1, \quad J_3 = \frac{1}{2}(x^2 - 2), \quad A_{12} = 1, \quad A_{13} = A_{23} = x - 1. \quad (2b)$$

The Hamiltonian can be derived from an integrable vertex model which results from fusion of the six-vertex model as we shall see below. It is called the integrable spin-one XXZ chain

as the derivation is similar to the way the standard spin-1/2 XXZ chain can be obtained from the six-vertex model. Let us mention some special cases for which the Hamiltonian simplifies. At $x = 2$, the Hamiltonian describes the $SU(2)$ -symmetric Babujian-Takhtajan spin chain [22, 23]. The point $x = 0$ is closely related to the so-called supersymmetric $t - J$ model [24]. In the limit $x \rightarrow \infty$ the spin chain becomes Ising-like, and hence very easy to analyse. Except for this last case, the diagonalisation of the Hamiltonian is a non-trivial problem. Nonetheless it can actually be done by using the Bethe ansatz.

In order to characterise the spin chain completely, we need to specify its boundary conditions. In this article, we are going to investigate the following, so-called diagonal twisted boundary conditions:

$$s_{N+1}^1 = \cos \phi s_1^1 - \sin \phi s_1^2, \quad s_{N+1}^2 = \sin \phi s_1^1 + \cos \phi s_1^2, \quad s_{N+1}^3 = s_1^3.$$

In fact, this corresponds to a simple rotation of the spin around axis 3 by the twist angle ϕ when going from site N to site 1. If $\phi = 0$, the boundary conditions are periodic and lead to a translation invariance of the spin chain $[H, S] = 0$. For non-zero twist angles, the system is however not invariant under translations. Yet, it is possible to introduce an appropriate notion of translation invariance by considering the modified translation operator $S' = S\Omega_N$. Here Ω_N is an operator acting on the last site, before the system is translated by one site. One verifies that $[H, S'] = 0$ for the boundary conditions given above, provided that

$$\Omega = \begin{pmatrix} e^{i\phi} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\phi} \end{pmatrix}. \quad (3)$$

Eventually, it is easy to see that for any twist angle the Hamiltonian commutes with the total magnetisation along the third axis, $[H, \Sigma] = 0$.

Supersymmetric sectors and singlets. It can be shown that the Hamiltonian (2) with the twists (3) has an exact lattice supersymmetry in certain subsectors of the Hilbert space. This means that there is an operator Q with $Q^2 = 0$ such that H can be written as the anticommutator

$$H = \{Q, Q^\dagger\}.$$

The special feature of Q in the present case is that it increases the number of sites by one whereas Q^\dagger decreases the length of the chain by one. The supersymmetry is therefore dynamic in the sense that the length of the chain changes through the action of its supercharges. The precise definition of Q and further technical can be found in [9], and a recent, very concise and general discussion of dynamic lattice supersymmetry in (super)spin chains in [25]. For our purposes, it is sufficient to know that the subsectors of V where the supersymmetry exists are the eigenspaces of the twisted translation operator S' with eigenvalue $(-1)^{N+1}$. Hence they depend on the twist angle. Indeed, the given eigenvalue implies that $(S')^N$ needs to act like the identity which leads to the condition that every vector in the subsector has to be an eigenvector of the operator $\Omega \otimes \Omega \otimes \dots \otimes \Omega$ with eigenvalue one. Using the explicit form (3) we find that this is possible if and only if $\phi\Sigma$ is an integer multiple of 2π . For example, this holds for periodic boundary conditions $\phi = 0$, or for the case $\phi = \pi$ provided that one restricts to subsectors where Σ is an even integer.

The existence of a supersymmetric structure on certain subspaces of V implies that within them the Hamiltonian is a positive definite operator. Its eigenvalues/energy levels are

bounded from below by zero. If a state $|\Phi\rangle$ with $H|\Phi\rangle = 0$ exists it is therefore automatically a ground state of the Hamiltonian in these subsectors. Such states are called supersymmetry singlets or simply zero-energy states, and annihilated by both the supercharge and its adjoint:

$$Q|\Phi\rangle = 0, \quad Q^\dagger|\Phi\rangle = 0.$$

The existence of such a supersymmetry singlet for (2) on chains of arbitrary length N , and arbitrary x was observed for chains of small length with twist angle $\phi = \pi$ in [26, 9]. Here, we prove this statement:

Theorem 2.1 *For any $N > 1$ and twist angle $\phi = \pi$ the Hamiltonian possesses a zero-energy state with zero total magnetisation in the subsector of the Hilbert space where the lattice supersymmetry exists.*

The proof relies on an explicit construction of the eigenstate. It is however important to stress that this might not be the only singlet. While this appears to be the case for most values of x , numerical studies of small systems suggests that there are special values for x , for instance $x = 0$, where additional zero-energy states occur. We address the uniqueness problem in a more general setting in conjecture 4.2. Moreover, we do not claim neither that the singlet is also the ground state when taking into account the full Hilbert space V . The exact diagonalisation of the Hamiltonian for small systems suggests that this might only be the case for large enough¹ x but a proof of this statement is beyond the scope of this article.

From supersymmetry to combinatorics. How does the zero-energy state look like? It is clear from the form of the Hamiltonian that we may choose its normalisation such that it is a polynomial in x with non-zero constant term. We expand it in the canonical basis according to

$$|\Phi(x)\rangle = \sum_{\sigma \in \{\uparrow, 0, \downarrow\}^N} \Phi_{\sigma_1 \dots \sigma_N}(x) |\sigma_1, \dots, \sigma_N\rangle.$$

The components $\Phi_{\sigma_1 \dots \sigma_N}(x)$ are all polynomials in x . For example, in the case of $N = 3$ sites we obtain:

$$\begin{aligned} \Phi_{\uparrow 0 \downarrow}(x) &= \Phi_{\downarrow 0 \uparrow}(x) = 1, \\ \Phi_{\downarrow \uparrow 0}(x) &= \Phi_{0 \downarrow \uparrow}(x) = \Phi_{\uparrow \downarrow 0}(x) = \Phi_{0 \uparrow \downarrow}(x) = -1, \\ \Phi_{000}(x) &= x. \end{aligned} \tag{4}$$

In [9] it was observed that some of these components, and in particular its square norm are given by generating functions for a certain type of weighted enumeration of alternating sign matrices. These are matrices with entries $-1, 0, 1$, and the rules that along each row and column the non-zero elements alternate in sign, with the first and last non-zero entry being 1. For instance, all 3×3 alternating sign matrices are given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

A closed formula for the number of $N \times N$ alternating sign matrices was conjectured by Mills, Robbins and Rumsey, and later proved by Zeilberger [27] (see [28] for an overview).

¹The author would like to thank Robert Weston and Junye Yang for pointing this out to him.

A short proof borrowing methods from quantum integrability was subsequently found by Kuperberg [29]. The enumeration problem can be refined as follows [30]: assign a weight t^k to all to every alternating matrix with exactly k entries -1 , and sum these weights for all $N \times N$ matrices. The result $A_N(t)$ is obviously a polynomial in t . For $N = 3$ we see from the matrices shown here above that $A_3(t) = 6 + t$. An explicit expression for $A_N(t)$ with arbitrary N in terms of a determinant formula can be found in [31, 32]. It implies in particular that $A_N(t)$ is a polynomial of degree $\lfloor (N-1)^2/4 \rfloor$.

Let us compare the example for $N = 3$ to the square norm of the vector (4). We use the *real* scalar product $\langle \cdot | \cdot \rangle$ on V , and therefore compute the norm $\|\Phi(x)\|^2 = \langle \Phi(x) | \Phi(x) \rangle$ according to

$$\|\Phi(x)\|^2 = \sum_{\sigma \in \{\uparrow, 0, \downarrow\}^N} \Phi_{\sigma_1 \dots \sigma_N}(x)^2.$$

For our example $N = 3$, we find

$$\|\Phi(x)\|^2 = 6 + x^2 = A_3(t = x^2).$$

This is not only coincidence for three sites. One verifies by exact diagonalisation of the Hamiltonian for small N that in a suitable normalisation, where all components are polynomials in x with integer coefficients, the square norm of $|\Phi(x)\rangle$ is indeed equal to $A_N(t = x^2)$.

To make this more precise, we have to fix the degree of the state as a polynomial in x in order to avoid redundancies. It is clear that we may restrict ourselves to impose the degree of a specific component. We choose $\Phi_{\uparrow \dots \uparrow \downarrow \dots \downarrow}(x)$ for even $N = 2n$, and $\Phi_{\uparrow \dots \uparrow 0 \downarrow \dots \downarrow}(x)$ for odd $N = 2n + 1$, and require them to be polynomials in x of degree $\lfloor (n-1)^2/4 \rfloor$ with the constant term being adjusted to

$$\Phi_{\underbrace{\uparrow \dots \uparrow}_n \underbrace{\downarrow \dots \downarrow}_n}(x) = n! + O(x), \quad \Phi_{\underbrace{\uparrow \dots \uparrow}_n \underbrace{0 \downarrow \dots \downarrow}_n}(x) = n! + O(x).$$

With this convention it is no longer possible to multiply the state with arbitrary polynomials which would only generate common (and redundant) factors of the components. We claim that with this normalisation scheme all components of the singlet are polynomials in x with integer coefficients. Moreover, the two special components are then given by

$$\Phi_{\underbrace{\uparrow \dots \uparrow}_n \underbrace{\downarrow \dots \downarrow}_n}(x) = \Phi_{\underbrace{\uparrow \dots \uparrow}_n \underbrace{0 \downarrow \dots \downarrow}_n}(x) = A_n(x^2), \quad (5)$$

and the square norm of the singlet takes the form

$$\|\Phi(x)\|^2 = A_N(x^2). \quad (6)$$

Here $A_N(t)$ is the generating function for the weighted enumeration of $N \times N$ alternating sign matrices with weight t per entry -1 .

The normalisation convention presented is different from the one in [9] where a restriction on the leading coefficient of $|\Phi(x)\rangle$ as a polynomial in x was imposed. The two conditions appear to be equivalent. However, the one stated here seems to be easier to prove.

3 The nineteen-vertex model

The spin-chain Hamiltonian (2) can be related to an integrable vertex model, the so-called nineteen-vertex model. The relation allows to study the spin chain with the help of tools from quantum integrability such as the algebraic Bethe ansatz. This is indeed the strategy which we pursue in order to prove the existence of the supersymmetry singlet. To this end, we need to recall the construction of the nineteen-vertex model through the so-called fusion procedure, and introduce furthermore the transfer matrices of the model with twisted boundary conditions and inhomogeneities.

Fusion. We use a parametrisation in terms of multiplicative spectral parameters, and make systematic use of the following abbreviation

$$[z] = z - z^{-1}.$$

The fusion procedure [33, 34] allows to construct iteratively solutions of the Yang-Baxter equation $R^{(m,n)}(z) \in \text{End}(\mathbb{C}^{m+1} \otimes \mathbb{C}^{n+1})$, $m, n = 1, 2, \dots$, starting from $m = n = 1$: they solve

$$R_{12}^{(m,n)}(z/w) R_{13}^{(m,p)}(z) R_{23}^{(n,p)}(w) = R_{23}^{(n,p)}(w) R_{13}^{(m,p)}(z) R_{12}^{(m,n)}(z/w), \quad (7)$$

on the product space $\mathbb{C}^{m+1} \otimes \mathbb{C}^{n+1} \otimes \mathbb{C}^{p+1}$. The indices i, j of $R_{ij}^{(m,n)}(z)$ label the factors of the tensor product which the R -matrices act on. The simplest case $m = n = 1$ corresponds R -matrix of the six-vertex model. Let us abbreviate the canonical basis of \mathbb{C}^2 by

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then, in the basis $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$ of $\mathbb{C}^2 \otimes \mathbb{C}^2$, we have

$$R^{(1,1)}(z) = \begin{pmatrix} [qz] & 0 & 0 & 0 \\ 0 & [z] & [q] & 0 \\ 0 & [q] & [z] & 0 \\ 0 & 0 & 0 & [qz] \end{pmatrix},$$

which solves (7) with $m = n = p = 1$. The matrix $R^{(1,1)}(z)$ degenerates at $z = q$ and $z = q^{-1}$ where it can be written in terms of the projectors P^+ and P^- onto the symmetric and antisymmetric subspaces of $\mathbb{C}^2 \otimes \mathbb{C}^2$:

$$R^{(1,1)}(z = q) = BP^+, \quad R^{(1,1)}(z = q^{-1}) = (-2[q])P^-,$$

with the diagonal matrix $B = \text{diag}([q^2], 2[q], 2[q], [q^2])$. This simple observation allows to construct $R^{(1,2)}(z)$ from the evaluation of the Yang-Baxter equation at the degeneration points. For example, setting $w = q^{-1}$ in the Yang-Baxter equation we see that $P_{23}^+ R_{12}^{(1,1)}(qz) R_{13}^{(1,1)}(z)$ leaves stable the symmetric subspace of the second and third factor in the tensor product. This can be extended to the following decomposition of a product of R -matrices:

$$M_{23} R_{12}^{(1,1)}(qz) R_{13}^{(1,1)}(z) M_{23}^{-1} = \begin{pmatrix} [qz] R_{1,(23)}^{(1,2)}(z) & 0 \\ * & [z][q^2 z] \end{pmatrix}.$$

The rows and columns of the matrix on the right-hand side are indexed by the symmetric and antisymmetric subspaces of $\mathbb{C}^2 \otimes \mathbb{C}^2 = \text{Sym}^2 \mathbb{C}^2 \oplus \bigwedge^2 \mathbb{C}^2$. Moreover, $M =$

$\text{diag}(1/\sqrt{[q^2]}, 1/\sqrt{2[q]}, 1/\sqrt{2[q]}, 1/\sqrt{[q^2]})$ is a diagonal matrix, whose introduction leads to a symmetric matrix $R^{(1,2)}(z)$. If we identify the basis vectors of $\text{Sym}^2 \mathbb{C}^2$ with the basis vectors of \mathbb{C}^3 according to

$$|\uparrow\rangle = |\uparrow\uparrow\rangle, \quad |0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), \quad |\downarrow\rangle = |\downarrow\downarrow\rangle,$$

then the $R^{(1,2)}(z)$ can be evaluated in compact form. Indeed, one shows that in the basis $\{|\uparrow\uparrow\rangle, |\uparrow 0\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow 0\rangle, |\downarrow\downarrow\rangle\}$ it is given by

$$R^{(1,2)}(z) = \begin{pmatrix} [q^2 z] & 0 & 0 & 0 & 0 & 0 \\ 0 & [qz] & 0 & \sqrt{[q][q^2]} & 0 & 0 \\ 0 & 0 & [z] & 0 & \sqrt{[q][q^2]} & 0 \\ 0 & \sqrt{[q][q^2]} & 0 & [z] & 0 & 0 \\ 0 & 0 & \sqrt{[q][q^2]} & 0 & [qz] & 0 \\ 0 & 0 & 0 & 0 & 0 & [q^2 z] \end{pmatrix}. \quad (8)$$

One verifies that it solves the Yang-Baxter equation (7) with $m = n = 1$, $p = 2$. In fact, it is known that $R^{(1,m)}(z)$ for all $m = 1, 2, \dots$ can be written down systematically in terms of the generators of the quantum group $U_q(\mathfrak{sl}_2)$ [35, 36]. The non-zero matrix elements of $R^{(1,2)}(q^{-1}z)$ can be interpreted as statistical weights for a vertex model on the square lattice whose horizontal edges are always oriented, whereas the vertical edges may be either oriented or not. The structure of (8) leaves us with the configurations of a mixed vertex model, shown in figure 1.

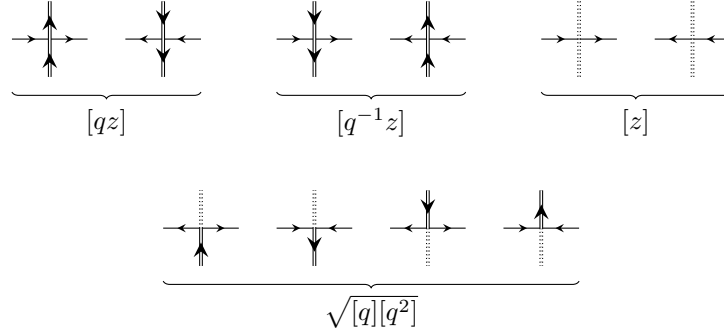


Figure 1: Vertices with non-zero weights of the mixed vertex model derived from the matrix $R^{(1,2)}(q^{-1}z)$. The spin components \uparrow and \downarrow are represented by simple arrows along the horizontal direction, oriented to the right and left respectively. Along the vertical direction, the spin components $\uparrow, 0, \downarrow$ are represented by upwards oriented lines, dotted lines without orientation or downwards oriented lines. In order to reconstruct the matrix elements, the pictures have to be read from south-west to north-east by following the lines. For example, from the first vertex in the bottom row we find $\langle \uparrow 0 | R^{(1,2)}(q^{-1}z) | \downarrow \uparrow \rangle = \sqrt{[q][q^2]}$.

The R -matrix of the nineteen-vertex model is constructed by applying once more the decomposition into symmetric, and antisymmetric subspaces to a product of R -matrices $R^{(1,2)}(z)$:

$$M_{12} R_{23}^{(1,2)}(z) R_{13}^{(1,2)}(q^{-1}z)^{(1,2)} M_{12}^{-1} = \begin{pmatrix} R_{(12),3}^{(2,2)}(z) & 0 \\ * & [q^{-1}z][q^2 z] \end{pmatrix}. \quad (9)$$

The non-zero matrix elements of $R(z) = R^{(2,2)}(z)$ correspond to the weights of the configurations of the integrable nineteen-vertex model which are shown in figure 2. Each edge of the lattice is either oriented (with two possibilities for the orientation) or without orientation. The configurations of the model are constrained by the rule that at the number of edges oriented towards any vertex is equal to the number of edges oriented away from it.

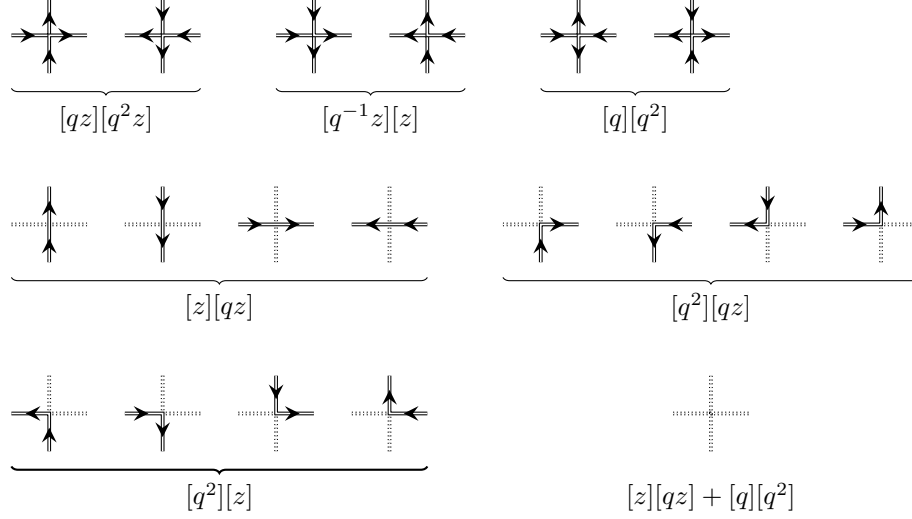


Figure 2: Vertices and weights of the nineteen-vertex model: the components \uparrow are represented by edges oriented to the right or top, 0 by dotted double edges without orientation, \downarrow by edges pointing down or to the left. In order to reconstruct the matrix elements of $R(z) = R^{(2,2)}(z)$ one reads the configurations from south-west to north-east by following the lines. For example : $\langle 0\uparrow | R(z) | \uparrow 0 \rangle = [q^2][qz]$.

Before proceeding to the construction of the transfer matrices of the nineteen-vertex model, we mention a few properties of the R -matrices. One verifies that the R -matrices $R^{(1,1)}(z)$, $R^{(1,2)}(z)$ and $R(z) = R^{(2,2)}(z)$ solve the Yang-Baxter equation (7) for all admissible combinations. The R -matrix of the nineteen-vertex model is of special interest to us. If we set its argument to $z = 1$ then it reduces up to a factor to the permutation operator P on $\mathbb{C}^3 \otimes \mathbb{C}^3$, i.e. the operator defined through $P(|v_1\rangle \otimes |v_2\rangle) = |v_2\rangle \otimes |v_1\rangle$:

$$R(z=1) = [q][q^2]P. \quad (10)$$

The R -matrix is symmetric and therefore we have $PR(z)P = R(z)$. It is occasionally useful to use the abbreviation $\check{R}(z) = PR(z) = R(z)P$ which reduces up to a factor to the identity at $z = 1$. Furthermore, one verifies by direct calculation that the so-called inversion relation holds

$$R(z)R(z^{-1}) = r(z)r(z^{-1}), \quad r(z) = [q/z][q^2z].$$

The fact that the left-hand side vanishes for certain values of z means that $R(z)$ cannot be invertible for all z . Like the six-vertex R -matrix $R^{(1,1)}(z)$ it has some degeneration points where it reduces to projectors on certain subspaces of $\mathbb{C}^3 \otimes \mathbb{C}^3$. A particular interesting point is $z = q^{-1}$ where its rank is one. We have

$$R(z = q^{-1}) = [q][q^2]|s\rangle\langle s|, \quad |s\rangle = |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle - |00\rangle. \quad (11)$$

Transfer matrices. Let us consider a transfer matrix built from the R -matrix $R^{(1,2)}(z)$:

$$T^{(1)}(z) = \text{tr}_a \left(\Omega_a^{(1)} R_{a,N}^{(1,2)}(q^{-1}z/w_j) \cdots R_{a,2}^{(1,2)}(q^{-1}z/w_2) R_{a,1}^{(1,2)}(q^{-1}z/w_1) \right). \quad (12)$$

Here w_j , $j = 1, \dots, N$ is an inhomogeneity parameter attached to the j -th factor of the N -fold tensor product V . If they all take the same value then the model is called homogeneous, otherwise inhomogeneous. The trace is taken over an auxiliary (horizontal) space \mathbb{C}^2 , labeled by a . The twist $\Omega^{(1)}$ denotes an operator acting on this auxiliary space. In this article, we consider the following diagonal case

$$\Omega^{(1)} = \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix}. \quad (13)$$

This choice yields a phase according to the direction of the spin on the auxiliary space before taking the trace.

The fusion procedure outlined above allows to construct the transfer matrix of the nineteen-vertex model $T^{(2)}(z)$ from $T^{(1)}(z)$. It is defined as

$$T^{(2)}(z) = \text{tr}_a \left(\Omega_a^{(2)} R_{a,N}(z/w_j) \cdots R_{a,2}(z/w_2) R_{a,1}(z/w_1) \right),$$

where the auxiliary space labeled by a is now \mathbb{C}^3 . With the twist from (13), the relation between the two types of transfer matrices is given by

$$T^{(2)}(z) = T^{(1)}(z)T^{(1)}(qz) + (-1)^{N+1} \prod_{j=1}^N [qw_j/z][q^2z/w_j]. \quad (14)$$

The derivation of this equation is based on the decomposition of the double trace over the two auxiliary spaces in the product $T^{(1)}(z)T^{(1)}(qz)$ into separate traces over the symmetric and antisymmetric subspaces of the product space, followed by the use of (9). $\Omega^{(2)}$ is determined from $\Omega^{(1)}$ by

$$\Omega^{(2)} = P^+(\Omega^{(1)} \otimes \Omega^{(1)})P^+ = \begin{pmatrix} e^{i\phi} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\phi} \end{pmatrix},$$

where P^+ is the projector onto the symmetric subspace of $\mathbb{C}^2 \otimes \mathbb{C}^2$. The result twist is therefore the same as in (3).

Let us now remind a few properties of the transfer matrices which are going to be relevant to our considerations. We emphasise here below the dependence on the inhomogeneity parameters $T^{(j)}(z) = T^{(j)}(z|w_1, \dots, w_N)$ for $j = 1, 2$. One of their most important features is that for a given set of inhomogeneities these matrices commute for different values of the spectral parameter:

$$[T^{(j)}(z|w_1, \dots, w_N), T^{(k)}(z'|w_1, \dots, w_N)] = 0,$$

for all z, z' and all choices for $j, k = 1, 2$. The commutation relations imply in particular that the eigenvectors of these matrices are independent of the spectral parameter z . Furthermore, it is sufficient to diagonalise $T^{(1)}(z|w_1, \dots, w_N)$ in order to compute both the spectrum and the eigenvectors of the $T^{(2)}(z|w_1, \dots, w_N)$. Another well-known consequence of the commutation relations is that it implies the existence of a large number of commuting

operators which contain at the homogeneous point $w_1 = \dots = w_N = 1$ both the twisted translation operator and the Hamiltonian of the spin-one XXZ chain (2). Indeed, using (10) one finds at the point $z = 1$:

$$S' = ([q][q^2])^{-N} T^{(2)}(z=1|w_1=1, \dots, w_N=1). \quad (15a)$$

The Hamiltonian is obtained from the logarithmic derivative at the same point

$$H = N + \frac{[q^2]}{2} \frac{d}{dz} \ln T^{(2)}(z|w_1=1, \dots, w_N=1) \Big|_{z=1}, \quad (15b)$$

provided that the parameter x used in section 2 is identified with

$$x = q + q^{-1}. \quad (15c)$$

Hence, the diagonalisation of the transfer matrices allows to recover the eigenvalues and -vectors of the spin-chain Hamiltonian.

4 Simple eigenvalue

In this section, we show that the inhomogeneous transfer matrix of the nineteen-vertex model with a diagonal twist possesses a simple eigenvalue for the twist angle $\phi = \pi$. The observation is astonishingly simple: we show that the Bethe equations of the model admit an explicit solution in terms of the inhomogeneity parameters. Taking the homogeneous limit, we observe that it leads to a zero eigenvalue of the Hamiltonian.

Algebraic Bethe ansatz. In the case of diagonal twists, the eigenvalues and -vectors of $T^{(1)}(z)$ and $T^{(2)}(z)$ can be constructed from the algebraic Bethe ansatz [20, 37]. The basic object is the monodromy matrix of the inhomogeneous model for N sites, given by

$$\mathcal{T}_a(z) = R_{a,N}^{(1,2)}(q^{-1}z/w_N) \cdots R_{a,1}^{(1,2)}(q^{-1}z/w_1).$$

It can be seen as a 2×2 matrix acting on the auxiliary space whose entries are operators on the Hilbert space V :

$$\mathcal{T}_a(z) = \begin{pmatrix} \mathcal{A}(z) & \mathcal{B}(z) \\ \mathcal{C}(z) & \mathcal{D}(z) \end{pmatrix}_a.$$

Comparing with (12) we conclude that

$$T^{(1)}(z|w_1, \dots, w_N) = \text{tr}_a \Omega_a^{(1)} \mathcal{T}_a(z) = e^{i\phi/2} \mathcal{A}(z) + e^{-i\phi/2} \mathcal{D}(z).$$

The entries of the monodromy matrix satisfy a number of quadratic relations which are a result of $R_{a_1 a_2}^{(1,1)}(z/w) \mathcal{T}_{a_1}(z) \mathcal{T}_{a_2}(w) = \mathcal{T}_{a_2}(w) \mathcal{T}_{a_1}(z) R_{a_1 a_2}^{(1,1)}(z/w)$, an immediate consequence of the Yang-Baxter equation (7). Here, we quote only a few of them relevant to our discussion:

$$\mathcal{A}(w) \mathcal{B}(z) = f(w, z) \mathcal{B}(z) \mathcal{A}(w) + g(w, z) \mathcal{B}(w) \mathcal{A}(z), \quad (16a)$$

$$\mathcal{D}(w) \mathcal{B}(z) = f(z, w) \mathcal{B}(z) \mathcal{D}(w) + g(z, w) \mathcal{B}(w) \mathcal{D}(z), \quad (16b)$$

$$[\mathcal{B}(z), \mathcal{B}(w)] = [\mathcal{C}(z), \mathcal{C}(w)] = 0, \quad (16c)$$

where we abbreviate

$$f(z, w) = [qw/z]/[w/z], \quad \text{and} \quad g(z, w) = [q]/[w/z].$$

The algebraic Bethe ansatz is based on a reference state (pseudo vacuum) $|\wedge\rangle$ defined through $\mathcal{C}(z)|\wedge\rangle = 0$ for any z . From the explicit form of $R^{(1,2)}(z)$ it is not difficult to see that it is given by the completely polarised state

$$|\wedge\rangle = |\uparrow\uparrow \cdots \uparrow\uparrow\rangle.$$

Furthermore, inspecting (8) one concludes that the action of the diagonal elements of the monodromy matrix on this state is very simple:

$$\mathcal{A}(z)|\wedge\rangle = a(z)|\wedge\rangle, \quad a(z) = \prod_{j=1}^N [qz/w_j], \quad \mathcal{D}(z)|\wedge\rangle = d(z)|\wedge\rangle, \quad d(z) = \prod_{j=1}^N [q^{-1}z/w_j].$$

The eigenstates of $T^{(1)}(z)$ are constructed by acting with the operator $\mathcal{B}(z)$ on the reference state. It is easy to see that $[\Sigma, \mathcal{B}(z)] = -\mathcal{B}(z)$: this operator flips spins and lowers the total magnetisation by one. An eigenvector with magnetisation $N - n$ is given by

$$|\Psi(z_1, \dots, z_n)\rangle = \prod_{j=1}^n \mathcal{B}(z_j)|\wedge\rangle. \quad (17)$$

As any two \mathcal{B} -operators commute according to (16) the expression is obviously symmetric in the z_1, \dots, z_n . These numbers are however not arbitrary. One shows that they lead to an eigenvector of $T^{(1)}(z)$ with eigenvalue

$$\theta^{(1)}(z|w_1, \dots, w_n) = e^{i\phi/2} a(z) \prod_{j=1}^n f(z, z_j) + e^{-i\phi/2} d(z) \prod_{j=1}^n f(z_j, z), \quad (18)$$

provided that the so-called Bethe equations hold. They are given by $a(z_k)/d(z_k) = e^{-i\phi} \prod_{j \neq k}^n f(z_j, z_k)/f(z_k, z_j)$, or more explicitly:

$$\prod_{j=1}^N \frac{[qz_k/w_j]}{[q^{-1}z_k/w_j]} = e^{-i\phi} \prod_{j \neq k}^n \frac{[qz_k/z_j]}{[q^{-1}z_k/z_j]}, \quad k = 1, \dots, n. \quad (19)$$

The derivation of these statements is based on the repeated application of the quadratic relations (16) in order to evaluate $\mathcal{A}(z)|\Psi(z_1, \dots, z_n)\rangle$ and $\mathcal{D}(z)|\Psi(z_1, \dots, z_n)\rangle$ [20]. The Bethe equations result from the elimination of so-called unwanted terms which occur when commuting $\mathcal{A}(z), \mathcal{D}(z)$ through the operators $\mathcal{B}(z_j)$ in (17). The eigenvalue is obtained after this when acting with $\mathcal{A}(z), \mathcal{D}(z)$ on the reference state. Because of the relation between $T^{(1)}(z)$ and $T^{(2)}(z)$ this procedure leads automatically to an eigenvector of the transfer matrix for the nineteen-vertex model. The eigenvalue can be computed from (14). In the homogeneous limit $w_1 = \dots = w_N = 1$ it allows therefore to diagonalise the spin-chain Hamiltonian.

Simple eigenvalue. Explicit solutions of the coupled algebraic equations (19) with finite N and n are rather difficult to obtain. Yet, in the present case there is a remarkably simple

solution, provided that the twist angle is chosen to be $\phi = \pi$ and the number of Bethe roots coincides with the number of sites $N = n$. It is given by

$$z_k = w_k, \quad k = 1, \dots, N. \quad (20)$$

Its insertion into (21) leads to a vanishing eigenvalue $\theta^{(1)}(z|z_1 = w_1, \dots, z_N = w_N) = 0$ for all z . We apply the fusion equation (14), and obtain the following result:

Theorem 4.1 *For the twist angle $\phi = \pi$ and any number of sites N , the transfer matrix of the nineteen-vertex model $T^{(2)}(z)$ possesses the eigenvalue*

$$\theta^{(2)}(z|w_1, \dots, w_N) = (-1)^{N+1} \prod_{j=1}^N [qw_j/z][q^2 z/w_j]. \quad (21)$$

This is in fact an inhomogeneous generalisation of the existence statements of theorem 2.1, appearing now as a mere corollary. Indeed, (21) holds for any choice of the inhomogeneity parameters and therefore by continuity also for the homogeneous model. We conclude therefore that if $w_1 = \dots = w_N = 1$ the transfer matrix possesses a Bethe eigenvector whose Bethe roots condense to a single point $z_1 = \dots = z_N = 1$. Using the relation between the transfer matrix and the spin-chain Hamiltonian given in (15) we conclude that the corresponding eigenstate has zero energy, and is an eigenvector of the twisted translation operator with eigenvalue $(-1)^{N+1}$. Hence, it is a ground state of the Hamiltonian in the subsector of the Hilbert space where the supersymmetry exists. Moreover, from $n = N$ we conclude that the resulting state has zero magnetisation. This completes the proof of theorem 2.1.

As for the homogeneous case discussed in section 2, we can at this point however not conclude that the eigenvalue non-degenerate even though the numerical diagonalisation for small system sizes suggests that this is the case:

Conjecture 4.2 *Except for possibly a finite number of values of the parameter q , the eigenvalue (21) is non-degenerate for all N .*

5 Properties of the eigenvector

In this section, we investigate the properties of the eigenvector of the transfer matrices $T^{(j)}(z|w_1, \dots, w_N)$, $j = 1, 2$ with the simple Bethe roots (20). It is given by

$$|\Psi(w_1, \dots, w_N)\rangle = \prod_{j=1}^N \mathcal{B}(w_j)|\wedge\rangle. \quad (22)$$

We start with analysing some simple properties of the state in section 5.1. In section 5.2 we use some of them in combination with Slavnov's formula in order to prove a multi-parameter sum rule for the eigenvector. We apply these results in section 5.3 in order to determine a simple component of the vector. The homogeneous point $w_1 = \dots = w_N = 1$ is studied in section 5.4.

5.1 Elementary properties

Here below, we present a simple graphical interpretation of the Bethe vector (22). Next, we establish an exchange relation which allows to permute its inhomogeneity parameters

through the action of R -matrices. We use it in order to show that the Bethe vector contains quite a few redundant factors. Hence we introduce a renormalised vector, and show that it solves a set of relations which are similar to the quantum Knizhnik-Zamolodchikov system. Eventually we determine its degree width as a Laurent polynomial in each variable.

Ground-state components as partition functions of the mixed vertex model.

The Bethe eigenvector (22) can be depicted graphically. In order to build it, we start the completely polarised state and act with the \mathcal{B} -operators N times. From the definition of the monodromy matrix we infer that these operators can be depicted as

$$\mathcal{B}(z) = \begin{array}{c} z \quad \leftarrow \quad \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} \quad \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} \quad \cdots \quad \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} \quad \rightarrow \end{array} \quad \begin{array}{c} \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} \\ w_1 \end{array} \quad \begin{array}{c} \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} \\ w_2 \end{array} \quad \cdots \quad \begin{array}{c} \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} \\ w_N \end{array} \quad (23)$$

where we indicated the parameters attached to the horizontal and vertical lines. The N -fold action is obtained from the vertical concatenation of these pictures which results in an $N \times N$ square grid whose horizontal lines have outpointing arrows, whereas the vertical lines along the bottom row have arrows inwards. The projection onto a basis vector fixes the boundary condition along the top row of the square, and leads to the following graphical representation for the component $\Psi_{\sigma_1 \sigma_2 \dots \sigma_N}(w_1, \dots, w_N) = \langle \sigma_1, \dots, \sigma_N | \Psi(w_1, \dots, w_N) \rangle$:

$$\Psi_{\sigma_1 \sigma_2 \dots \sigma_N}(w_1, \dots, w_N) = \begin{array}{c} \begin{array}{c} \sigma_1 \quad \sigma_2 \quad \cdots \quad \sigma_N \\ \begin{array}{|c|c|c|c|} \hline \text{---} & \text{---} & \text{---} & \text{---} \\ \hline \end{array} \\ \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} \end{array} \quad \begin{array}{c} \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} \\ w_N \end{array} \quad \begin{array}{c} \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} \\ \vdots \end{array} \quad \begin{array}{c} \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} \\ w_2 \end{array} \quad \begin{array}{c} \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} \\ w_1 \end{array} \end{array} \quad \begin{array}{c} \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} \\ w_1 \end{array} \quad \begin{array}{c} \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} \\ w_2 \end{array} \quad \cdots \quad \begin{array}{c} \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} \\ w_N \end{array} \quad (24)$$

Any component of the Bethe vector coincides therefore with a partition function of the mixed vertex model on such an $N \times N$ square. The Boltzmann weight for a vertex in row i and column j is given by $R^{(1,2)}(q^{-1}w_i/w_j)$, the weight of a configuration by the product of its vertex weights, and the partition function by the sum over the weights of all configurations compatible with the boundary conditions. Because of the commutativity of the \mathcal{B} -operators for different arguments (16), the vertical arrangement of the parameters w_1, \dots, w_N is completely arbitrary. The graphical representation will be useful in order to compute a certain simple component of the eigenvector, and study the point $w_1 = \dots = w_N = 1$.

Exchange relation. Let us now analyse how to permute the arguments of the Bethe vector (22). Notice that even though a generic Bethe state (17) is symmetric in the Bethe roots we cannot conclude that $|\Psi(w_1, \dots, w_N)\rangle$ is symmetric in its arguments, because the \mathcal{B} -operators carry an explicit dependence *and* implicit on these parameters. Nonetheless, it is possible to derive an exchange relation. To this end, we examine the action of the matrix $\check{R}(z)$ on the vector (22). It is useful to study the commutation relation between the monodromy matrix and the R -matrix of the nineteen-vertex model. Writing $\mathcal{T}_a(z) = \mathcal{T}_a(z|w_1, \dots, w_N)$ we obtain

$$\mathcal{T}_a(z|\dots, w_{j+1}, w_j, \dots) \check{R}_{j,j+1}(w_j/w_{j+1}) = \check{R}_{j,j+1}(w_j/w_{j+1}) \mathcal{T}_a(z|\dots, w_j, w_{j+1}, \dots),$$

for $j = 1, \dots, N-1$. Indeed, this is easily proved by writing both sides as a product of R -matrices, and applying the Yang-Baxter equation. As $\check{R}_{j,j+1}(w_j/w_{j+1})$ does not act on the auxiliary space, this relation holds for all matrix elements of the monodromy matrix, in particular for $\mathcal{B}(z) = \mathcal{B}(z|w_1, \dots, w_N)$. Taking into account that $\check{R}_{j,j+1}(z)|\wedge\rangle = [qz][q^2z]|\wedge\rangle$ we find thus from (22) the relation

$$\check{R}_{j,j+1}\left(\frac{w_j}{w_{j+1}}\right)|\Psi(\dots, w_j, w_{j+1}, \dots)\rangle = \left[\frac{qw_j}{w_{j+1}}\right]\left[\frac{q^2w_j}{w_{j+1}}\right]|\Psi(\dots, w_{j+1}, w_j, \dots)\rangle. \quad (25)$$

Renormalised eigenvector. We show now that (25) implies that the eigenvector is proportional to a product of elementary factors which are common to all its components:

Proposition 5.1 *All components of the vector $|\Psi(w_1, \dots, w_N)\rangle$ are proportional to the product $\prod_{1 \leq j < k \leq N} [qw_j/w_k]$.*

Proof: We evaluate the action of the transfer matrix $T^{(2)}(w = w_j|w_1, \dots, w_N)$ on the vector in two ways. On the one hand, we may use the fact that (22) is an eigenvector with eigenvalue (21). On the other hand, at $w = w_j$ the transfer matrix becomes a product of R -matrices and the twisted translation operator:

$$\begin{aligned} T^{(2)}(w_j|w_1, \dots, w_N) &= [q][q^2]\check{R}_{j-1,j}\left(\frac{w_j}{w_{j-1}}\right) \cdots \check{R}_{1,2}\left(\frac{w_j}{w_1}\right) S' \\ &\quad \times \check{R}_{N-1,N}\left(\frac{w_j}{w_N}\right) \cdots \check{R}_{j,j+1}\left(\frac{w_j}{w_{j+1}}\right). \end{aligned} \quad (26)$$

The result is sometimes called a scattering operator as its effect is to drag the inhomogeneity at position j through all others in a cyclic manner.

We equate both expressions, use the exchange relation (25), and find after elimination of some trivial factors the following relation for all $j = 1, \dots, N-1$:

$$\begin{aligned} \prod_{k=j+1}^N [qw_j/w_k] \prod_{1 \leq k \leq j-1} \check{R}_{k,k+1}(w_j/w_k) S' |\Psi(w_1, \dots, w_{j-1}, w_{j+1}, \dots, w_N, w_j)\rangle \\ = (-1)^{N+1} \prod_{k=1, k \neq j}^N [qw_k/w_j] \prod_{k=1}^{j-1} [q^2w_j/w_k] |\Psi(w_1, \dots, w_N)\rangle. \end{aligned} \quad (27)$$

The arrow \curvearrowright indicates that the product over the R -matrices is taken in reverse order. This equation states the equality of two vectors which are symmetric Laurent polynomials in each variable. In particular, they have the same zeroes, and the same degree for their leading and trailing terms. We conclude our eigenvector on the right-hand side is proportional to the first factor of the left-hand side

$$\prod_{k=j+1}^N [qw_j/w_k] \quad \text{for all } j = 1, \dots, N-1,$$

and hence to the product of all these factors. This proves the claim. \square

The preceding proposition suggests to divide out the redundant factors. Therefore, we introduce the renormalised vector

$$|\tilde{\Psi}(w_1, \dots, w_N)\rangle = \left(([q][q^2])^{N/2} \prod_{1 \leq j < k \leq N} [qw_j/w_k] \right)^{-1} |\Psi(w_1, \dots, w_N)\rangle. \quad (28)$$

The additional division by $([q][q^2])^{N/2}$ is chosen for convenience in order to remove common multiplicative constants which come from the spin flips which the \mathcal{B} -operator induces.

Recurrence. The vector $|\tilde{\Psi}(w_1, \dots, w_N)\rangle$ obeys an exchange relation, too. Indeed, comparing with (25) we obtain for all $j = 1, \dots, N-1$

$$\check{R}_{j,j+1} \left(\frac{w_j}{w_{j+1}} \right) |\tilde{\Psi}(\dots, w_j, w_{j+1}, \dots)\rangle = \left[\frac{qw_{j+1}}{w_j} \right] \left[\frac{q^2 w_j}{w_{j+1}} \right] |\tilde{\Psi}(\dots, w_{j+1}, w_j, \dots)\rangle, \quad (29a)$$

whose right-hand side differs slightly from (25). Moreover, the renormalised vector transforms covariantly under cyclic shifts. Indeed, using (27) with $j = 1$, we obtain after a slight redefinition of the variables the equation

$$S' |\tilde{\Psi}(w_1, \dots, w_{N-1}, w_N)\rangle = (-1)^{N+1} |\tilde{\Psi}(w_N, w_1, \dots, w_{N-1})\rangle. \quad (29b)$$

The equations (29a) and (29b) are akin to the so-called quantum Knizhnik-Zamolodchikov system which was used in order to study similar problems. They allow to shuffle the inhomogeneity parameters around. Moreover, they allow to establish a relation between the vectors for lengths N and $N-2$. This can be seen as follows: if we replace $w_j \rightarrow q^{-1}w_j$ and $w_{j+1} \rightarrow w_j$ then the exchange relation becomes

$$|\tilde{\Psi}(\dots, w_j, q^{-1}w_j, \dots)\rangle = ([q][q^2])^{-1} \check{R}_{j,j+1}(q^{-1}) |\tilde{\Psi}(\dots, q^{-1}w_j, w_j, \dots)\rangle.$$

For $z = q^{-1}$ the R -matrix of the nineteen-vertex model is of rank one as was shown in (11): we have $\check{R}(q^{-1}) = [q][q^2]|s\rangle\langle s|$ where $|s\rangle = |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle - |00\rangle$. We conclude that if $w_{j+1} = q^{-1}w_j$ then the vector is proportional to $|s\rangle$ on sites $j, j+1$. As we are free to permute the inhomogeneity parameters with the help of the exchange relation, it is sufficient to consider the case $j = 1$. Taking advantage of the explicit form of the renormalised eigenvector in terms of \mathcal{B} -operators, one finds after a calculation the following relation:

$$|\tilde{\Psi}(w_1, q^{-1}w_1, w_3, \dots, w_N)\rangle = (-1)^N [q] \left(\prod_{j=3}^N \left[\frac{qw_1}{w_j} \right] \left[\frac{q^2 w_j}{w_1} \right] \right) |s\rangle \otimes |\tilde{\Psi}(w_3, \dots, w_N)\rangle.$$

This equation relates the eigenstates at N and $N-2$ through specialisation of a pair of inhomogeneity parameters.

Degree width. Let us consider a simple example for the components of the renormalised vector $|\tilde{\Psi}\rangle$. Because of covariance under cyclic shifts we indicate the expressions only for representatives. For $N = 3$ we find the expressions

$$\begin{aligned} \tilde{\Psi}_{\uparrow 0 \downarrow}(w_1, w_2, w_3) &= \tilde{\Psi}_{\downarrow 0 \uparrow}(w_1, w_2, w_3) = [q][qw_2/w_1][qw_3/w_2], \\ \tilde{\Psi}_{000}(w_1, w_2, w_3) &= [q]^2[q^2] + [w_2/w_1][w_3/w_1][w_3/w_2]. \end{aligned}$$

The components are thus centred Laurent polynomials in each variable w_j . The degree width of a Laurent polynomial is the difference of the degrees of its leading and trailing terms: for instance, $\sum_{j=-m}^n a_j w^j$ with $a_{-m}, a_n \neq 0$ has degree width $m+n$ with respect to w . From our example for $N=3$, we see that the degree width of the eigenvector in a given variable varies from component to component. The degree width of the vector $|\tilde{\Psi}(w_1, \dots, w_N)\rangle$ in w_j is the maximum degree width of its components.

Proposition 5.2 *The degree width of the vector $|\tilde{\Psi}(w_1, \dots, w_N)\rangle$ in each variable is $2(N-1)$.*

Proof: The degree width is obviously an additive quantity under multiplication of Laurent polynomials. Consider the operator $\mathcal{B}(z|w_1, \dots, w_N)$. It is a centred Laurent polynomial in all its arguments of degree width $2N$ in z , and 2 in w_k for all $k=1, \dots, N$. If we specify however $z=w_j$ for some j then the monodromy matrix, and thus $\mathcal{B}(w_j|w_1, \dots, w_N)$, contains an R -matrix $R_{a,j}^{(1,2)}(q^{-1})$. Therefore the degree width of $\mathcal{B}(w_j|w_1, \dots, w_N)$ is $2(N-1)$ in w_j , and 2 in all other w_k , $k \neq j$. The construction (22) implies thus that $|\Psi(w_1, \dots, w_N)\rangle$ has degree width $4(N-1)$ in each variable because of the additivity property. In (28) we divide out a multi-variable Laurent polynomial which is obviously of degree width $2(N-1)$ in each of its variables. Hence, in any w_j the degree width of $|\tilde{\Psi}(w_1, \dots, w_N)\rangle$ is $4(N-1) - 2(N-1) = 2(N-1)$. \square

It is possible to work out relations between components in the limits where $w_j \rightarrow 0$ or $w_j \rightarrow \infty$, i.e. the highest components of the leading and trailing parts of the vector as a Laurent polynomial in w_j . These follow straightforwardly from the graphical representation of the Bethe vector and lead to another recurrence relation. The key observation is to notice that the R -matrix $R^{(1,2)}(z)$ is diagonal at leading order when its argument is sent to zero or infinity. This allows to delete the j -th row and column from the picture (24). After proper normalisation and some calculation, one finds the following result:

Proposition 5.3 *In the limit of infinite or zero spectral parameter w_j the components of the renormalised Bethe vector for N sites reduce at leading order to the components for $N-1$ sites according to*

$$\begin{aligned} \lim_{w_j \rightarrow \infty} w_j^{-(N-1)} \tilde{\Psi} \dots \sigma_{j-1} \sigma_j \sigma_{j+1} \dots (w_1, \dots, w_j, \dots, w_N) \\ = (-1)^{N-j} \delta_{\sigma_j, 0} \left(\prod_{k \neq j}^N w_k^{-1} \right) \tilde{\Psi} \dots \sigma_{j-1} \sigma_{j+1} \dots (w_1, \dots, w_{j-1}, w_{j+1}, \dots, w_N), \end{aligned}$$

and

$$\begin{aligned} \lim_{w_j \rightarrow 0} w_j^{N-1} \tilde{\Psi} \dots \sigma_{j-1} \sigma_j \sigma_{j+1} \dots (w_1, \dots, w_j, \dots, w_N) \\ = (-1)^{j-1} \delta_{\sigma_j, 0} \left(\prod_{k \neq j}^N w_k \right) \tilde{\Psi} \dots \sigma_{j-1} \sigma_{j+1} \dots (w_1, \dots, w_{j-1}, w_{j+1}, \dots, w_N), \end{aligned}$$

where $\delta_{ab} = 1$ for $a=b$, and 0 otherwise.

5.2 Scalar products and the square norm

The completely explicit nature of the Bethe roots (20) implies that all components of the eigenvectors $|\Psi(w_1, \dots, w_N)\rangle$ and $|\tilde{\Psi}(w_1, \dots, w_N)\rangle$ can in principle be computed in finite

size. It is desirable to develop a systematic practical scheme to do this for arbitrary finite N . We leave this to a future investigation. In this article we evaluate only a simple component which turns out to be intimately related to the square norm of the vector. Hence, we discuss first the square norm in this section.

A natural generalisation of the square norm of the eigenstates for the spin-one XXZ chain is the infinite-cylinder partition function (see [18] for an explanation of this interpretation) :

$$\begin{aligned} Z(w_1, \dots, w_N) &= \langle \tilde{\Psi}(w_1^{-1}, \dots, w_N^{-1}) | \tilde{\Psi}(w_1, \dots, w_N) \rangle \\ &= \sum_{\sigma \in \{\uparrow, 0, \downarrow\}^N} \tilde{\Psi}_{\sigma_1 \dots \sigma_N}(w_1^{-1}, \dots, w_N^{-1}) \tilde{\Psi}_{\sigma_1 \dots \sigma_N}(w_1, \dots, w_N). \end{aligned}$$

As in the homogeneous case, we use the *real* scalar product on V . In order to evaluate $Z(w_1, \dots, w_N)$ explicitly, we rewrite it in terms of the elements \mathcal{B}, \mathcal{C} of the monodromy matrix, and use the theory of scalar products of the algebraic Bethe ansatz. To this end, we use the fact that under transposition the operator \mathcal{B} can be transformed to \mathcal{C} according to

$$\mathcal{B}(z^{-1} | w_1^{-1}, \dots, w_N^{-1})^t = (-1)^{N-1} \mathcal{C}(z | w_1, \dots, w_N).$$

This is a direct consequence of the so-called crossing symmetry of $R^{(1,2)}(z)$: the transpose with respect to the second space it acts on can be written as

$$R^{(1,2)}(z)^{t_2} = (\sigma^2 \otimes 1) R^{(1,2)}(q^{-2} z^{-1})^{t_2} (\sigma^2 \otimes 1), \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Combining this with the definition of the renormalised Bethe vector (28) we obtain

$$Z(w_1, \dots, w_N) = \frac{\langle \wedge | \prod_{j=1}^N \mathcal{C}(w_j | w_1, \dots, w_N) \prod_{j=1}^N \mathcal{B}(w_j | w_1, \dots, w_N) | \wedge \rangle}{(-1)^N [q^2]^N \prod_{j,k=1}^N [q^{-1} w_j / w_k]}. \quad (30)$$

Scalar products of this type can be computed with the help of determinant formulae thanks to Slavnov's formula [21]. The original proof was given for periodic boundary conditions but the generalisation to twisted boundary conditions is immediate :

Theorem 5.4 (Slavnov's formula) *Let $n > 0$ and consider a solution z_1, \dots, z_n to the Bethe ansatz equations (19) for N sites and a set of arbitrary numbers ζ_1, \dots, ζ_n . Then the scalar product*

$$S_n = \langle \wedge | \prod_{j=1}^n \mathcal{C}(z_j) \prod_{j=1}^n \mathcal{B}(\zeta_j) | \wedge \rangle$$

is given by

$$S_n = \prod_{j=1}^n d(z_j) d(\zeta_j) \prod_{1 \leq k < j \leq n} g(z_j, z_k) g(\zeta_k, \zeta_j) \prod_{j,k=1}^n \frac{f(z_j, \zeta_k)}{g(z_j, \zeta_k)} \det M,$$

where $M = (M_{jk})$ is an $n \times n$ matrix with entries

$$M_{jk} = e^{-i\phi} \frac{g(z_j, \zeta_k)^2}{f(z_j, \zeta_k)} - \frac{g(\zeta_k, z_j)^2}{f(\zeta_k, z_j)} \frac{a(\zeta_k)}{d(\zeta_k)} \prod_{m=1}^n \frac{f(\zeta_k, z_m)}{f(z_m, \zeta_k)}, \quad j, k = 1, \dots, n.$$

For our problem the functions $a(z), d(z), g(z, w), f(z, w)$ were defined in section 4. The result is in fact very strong as it does not depend on their precise form but just a few analyticity properties.

The expression simplifies considerably if we specify for $\phi = \pi, n = N$ the Bethe roots to (20), and make use of the definition of $a(z), d(z)$. In this case, we obtain after some algebra the expression

$$S_N = (-1)^N \left(\prod_{j=1}^N d(w_j) \right) Z_{\text{IK}}(\zeta_1, \dots, \zeta_N; w_1, \dots, w_N). \quad (31)$$

Here $Z_{\text{IK}}(\zeta_1, \dots, \zeta_N; z_1, \dots, z_N)$ is given by the so-called Izergin-Korepin determinant formula [38]:

$$Z_{\text{IK}}(\zeta_1, \dots, \zeta_N; w_1, \dots, w_N) = \frac{\prod_{j,k=1}^N \mathbf{a}(\zeta_j/w_k) \mathbf{b}(\zeta_j/w_k)}{\prod_{j < k} [\zeta_j/\zeta_k][w_k/w_j]} \det_{j,k=1, \dots, N} \left(\frac{\mathbf{c}(\zeta_j/w_k)}{\mathbf{a}(\zeta_j/w_k) \mathbf{b}(\zeta_j/w_k)} \right). \quad (32)$$

The functions $\mathbf{a}(z), \mathbf{b}(z), \mathbf{c}(z)$ are the statistical weights for the first, second, and third group of vertices of a six-vertex model, shown in figure 3. The Izergin-Korepin determinant

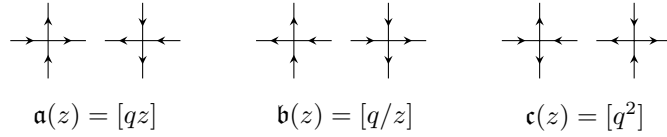


Figure 3: Vertex configurations of the six-vertex model. A vertex of the first, second or third group has weight $\mathbf{a}(z)$, $\mathbf{b}(z)$ or $\mathbf{c}(z)$ respectively.

(32) is the partition function of the inhomogeneous six-vertex model on an $N \times N$ square lattice with so-called domain wall boundary conditions as illustrated on figure 4. At the top and bottom row of the square all arrows are outgoing, whereas they are ingoing at its left- and rightmost column. The weight of a vertex in row j and column k is chosen with spectral parameter $z = \zeta_j/w_k$ for $j, k = 1, \dots, N$. The weight of a configuration is the product of the weights for each of its vertices.

If we set $\zeta_j = w_j$ for all $j = 1, \dots, N$, and combine the expression of the partition function (30) with our findings, then we obtain the following closed form:

Proposition 5.5 *The renormalised Bethe vector $|\tilde{\Psi}(w_1, \dots, w_N)\rangle$ satisfies the following sum rule:*

$$Z(w_1, \dots, w_N) = [q^2]^{-N} Z_{\text{IK}}(w_1, \dots, w_N; w_1, \dots, w_N).$$

Apart from leading to an explicit formula for the square norm of $|\tilde{\Psi}\rangle$, the identification (31) has another interesting consequence. Notice that if we choose ζ_1, \dots, ζ_N to be a solution of the Bethe equations with $\phi = \pi, n = N$ which leads to another eigenvalue, then the scalar product has to be zero. This implies that all such solutions need to solve $Z_{\text{IK}}(\zeta_1, \dots, \zeta_N; w_1, \dots, w_N) = 0$ (in addition to the Bethe equations): they lie in the algebraic variety defined by the zeroes of the Izergin-Korepin determinant in the first set of variables. It might be interesting to study the nature of these points on the variety.

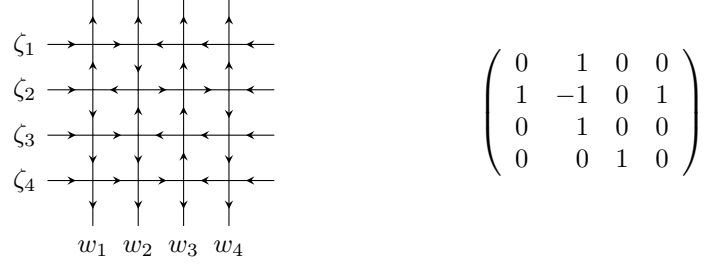


Figure 4: *Left*: $N \times N$ square lattice with domain-wall boundary conditions for $N = 4$. The weight of a vertex in row j and column k is a function of the spectral parameters ζ_j and w_k . *Right*: The corresponding alternating sign matrix.

5.3 Simple components

In this section we use the closed expressions for scalar products involving the vector $|\tilde{\Psi}(w_1, \dots, w_N)\rangle$ in order to determine the components

$$\tilde{\Psi}_{\uparrow \dots \uparrow \downarrow \dots \downarrow}^{(w_1, \dots, w_{2n})} \quad \text{and} \quad \tilde{\Psi}_{\uparrow \dots \uparrow 0 \downarrow \dots \downarrow}^{(w_1, \dots, w_{2n+1})}$$

for even length $N = 2n$ and odd length $N = 2n + 1$ respectively. They are the most simple components in the sense that they can be evaluated as a product of monomials and the Izergin-Korepin determinant.

Even lengths. We start with even $N = 2n$. We claim that the component is given by the following matrix element for a system of length $n = N/2$:

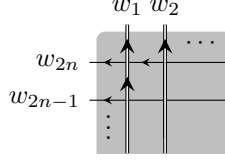
$$\tilde{\Psi}_{\uparrow \dots \uparrow \downarrow \dots \downarrow}^{(w_1, \dots, w_{2n})} = \frac{\prod_{j=1}^{2n} \prod_{k=1}^n [q^{-1} w_j / w_k]}{([q][q^2])^n \prod_{j < k}^{2n} [q w_j / w_k]} \langle \vee | \prod_{j=1}^{2n} \mathcal{B}(w_j | w_{n+1}, \dots, w_{2n}) | \wedge \rangle. \quad (33)$$

Here $|\vee\rangle = |\downarrow \dots \downarrow\rangle$ is the spin-reversed Bethe reference state $|\wedge\rangle = |\uparrow \dots \uparrow\rangle$ for n sites. This surprising reduction of the system size is most easily seen from the graphical representation for the corresponding Bethe vector component $\Psi_{\uparrow \dots \uparrow \downarrow \dots \downarrow}^{(w_1, \dots, w_{2n})}$. Let us use the recipe of section 5.1, and write it down graphically as the left-hand side of the following equation:

$$= \prod_{j=1}^{2n} \prod_{k=1}^n \left[\frac{w_j}{q w_k} \right] \times \quad (34)$$

In fact, the shaded region can be replaced by a simple product of Boltzmann weights. To see this, we inspect the vertex located at the upper-left corner of the picture: it has two

outgoing arrows. The only possibility for it to have a non-vanishing Boltzmann weight is that its lower and right edge are ingoing. This gives the following configuration:



We see that the vertices in the immediate neighbourhood of the corner have now two outgoing arrows, and hence their remaining edges need to be ingoing: this situation propagates, and allows to peel off row by row (or column by column) from the shaded region and replace the vertices by their weights $[q^{-1}w_j/w_k]$, $j = 1, \dots, 2n$, $k = 1, \dots, n$. This leads to the factor of the right-hand side in (34). Using then the graphical representation of $\mathcal{B}(w|w_{n+1}, \dots, w_{2n})$ as shown in (23) leads, after correct normalisation, to the expression (33).

The next step consists of converting (33) into a scalar product which can be computed with the help of Slavnov's formula. To this end, we need a short digression on the spin-reversal operator \mathfrak{R} . For length N , it acts like the matrix

$$\mathfrak{R} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}^{\otimes N}.$$

It is not very difficult to show that the entries of the monodromy matrix can be related by spin reversal according to

$$\mathfrak{R}\mathcal{A}(z|w_1, \dots, w_N) = \mathcal{D}(z|w_1, \dots, w_N)\mathfrak{R}, \quad \mathfrak{R}\mathcal{B}(z|w_1, \dots, w_N) = \mathcal{C}(z|w_1, \dots, w_N)\mathfrak{R}$$

Hence the transfer matrices $T^{(j)}(w|w_1, \dots, w_N)$, $j = 1, 2$, with twist angle $\phi = \pi$ satisfy the (anti)commutation relations

$$\mathfrak{R}T^{(j)}(z|w_1, \dots, w_N) = (-1)^j T^{(j)}(z|w_1, \dots, w_N)\mathfrak{R}.$$

Interestingly, the anticommutation relation between $T^{(1)}(z)$ and \mathfrak{R} implies that for any eigenstate $|\Psi\rangle$ with $T^{(1)}(z)|\Psi\rangle = \theta^{(1)}(z)|\Psi\rangle$ the state $\mathfrak{R}|\Psi\rangle$ is also an eigenstate of $T^{(1)}(z)$ with eigenvalue $-\theta^{(1)}(z)$. Hence, all their simultaneous eigenvectors must be annihilated by $T^{(1)}(z)$. Unfortunately, this does not imply that a given eigenvector with $\theta^{(1)}(z) = 0$, is also an eigenvector of \mathfrak{R} . To proceed, we make use of conjecture 4.2 which claims that the eigenspace of $\theta^{(1)}(z) = 0$ is one-dimensional for any N (except for possibly a finite number of values for q). Therefore, our (renormalised) Bethe state needs to be an eigenvector of the spin-reversal operator for any N . The recurrence relations of proposition 5.3 imply that the eigenvalue needs to be the same for all N . For $N = 1$ we have simply $|\tilde{\Psi}(w_1)\rangle = |0\rangle$ which is invariant under spin reversal: $\mathfrak{R}|\tilde{\Psi}(w_1)\rangle = |\tilde{\Psi}(w_1)\rangle$. Hence, for arbitrary N we obtain the relation

$$\mathfrak{R}|\tilde{\Psi}(w_1, \dots, w_N)\rangle = |\tilde{\Psi}(w_1, \dots, w_N)\rangle,$$

and of course the same equation for $|\Psi(w_1, \dots, w_N)\rangle$. Written out explicitly in terms of the operators \mathcal{B} and \mathcal{C} we find therefore

$$\prod_{j=1}^N \mathcal{C}(w_j|w_1, \dots, w_N)|\vee\rangle = \prod_{j=1}^N \mathcal{B}(w_j|w_1, \dots, w_N)|\wedge\rangle. \quad (35)$$

Let us now come back to the evaluation of our component through (33). We use the transposed version of (35) for n sites in order to convert half of the \mathcal{B} -operators into \mathcal{C} 's:

$$\begin{aligned}\tilde{\Psi}_{\uparrow\cdots\uparrow\downarrow\cdots\downarrow}(w_1, \dots, w_{2n}) &= \frac{\prod_{j=1}^{2n} \prod_{k=1}^n [q^{-1}w_j/w_k]}{([q][q^2])^n \prod_{j < k}^{2n} [qw_j/w_k]} \\ &\times \langle \wedge | \prod_{j=n+1}^{2n} \mathcal{C}(w_j | w_{n+1}, \dots, w_{2n}) \prod_{j=1}^n \mathcal{B}(w_j | w_{n+1}, \dots, w_{2n}) | \wedge \rangle.\end{aligned}$$

We see thus appear a typical scalar product which can be evaluated from (31) for a system of length n . Hence, the component can be written in terms of the Izergin-Korepin determinant. Some of the pre-factors cancel out and the final result takes the compact form

$$\begin{aligned}\tilde{\Psi}_{\uparrow\cdots\uparrow\downarrow\cdots\downarrow}(w_1, \dots, w_{2n}) &= \left(\frac{[q]}{[q^2]} \right)^n \prod_{1 \leq j < k \leq n} \left[\frac{qw_k}{w_j} \right] \prod_{n+1 \leq j < k \leq 2n} \left[\frac{qw_k}{w_j} \right] \\ &\times Z_{\text{IK}}(w_1, \dots, w_n; w_{n+1}, \dots, w_{2n}).\end{aligned}\quad (36)$$

Odd lengths. For $N = 2n + 1$ the component $\tilde{\Psi}_{\uparrow\cdots\uparrow 0 \downarrow \cdots \downarrow}(w_1, \dots, w_{2n+1})$ can be obtained from the result at even length, the known degree width of the vectors and the exchange relation (29a). The latter can be projected on spin configurations which allows us to investigate the dependence of the components on their parameters. For instance, it is easily shown that

$$\left[\frac{qw_{j+1}}{w_j} \right] \tilde{\Psi}_{\uparrow\cdots\uparrow\uparrow\cdots}(\dots, w_j, w_{j+1}, \dots) = \left[\frac{qw_j}{w_{j+1}} \right] \tilde{\Psi}_{\uparrow\cdots\uparrow\uparrow\cdots}(\dots, w_{j+1}, w_j, \dots).$$

This implies in particular that any component with the pattern $\uparrow\uparrow$ at positions $j, j+1$ is proportional to $[qw_j/w_{j+1}]$ times a centred Laurent polynomial which is symmetric under the exchange of w_j and w_{j+1} . Another simple consequence of the exchange relation is

$$\tilde{\Psi}_{\uparrow\cdots\uparrow 0 \cdots}(\dots, w_j, w_{j+1}, \dots) = \left[\frac{qw_{j+1}}{w_j} \right] \tilde{\Upsilon}_{\uparrow\cdots\uparrow 0 \cdots}(\dots, w_j, w_{j+1}, \dots),$$

where $\tilde{\Upsilon}_{\uparrow\cdots\uparrow 0 \cdots}(\dots, w_j, w_{j+1}, \dots)$ is some centred Laurent polynomial in its arguments. Combining this with the symmetry property of the arguments within a string $\uparrow \cdots \uparrow$ allows to conclude that the component $\tilde{\Psi}_{\uparrow\cdots\uparrow 0 \downarrow \cdots \downarrow}(w_1, \dots, w_{2n+1})$ is actually proportional to $[qw_{n+1}/w_j]$ for all $j = 1, \dots, n$, and hence to their product. A similar relation is found when analysing the dependence on the parameters to the right of the site $n+1$. Factor exhaustion leads thus to the following form for our component:

$$\begin{aligned}\tilde{\Psi}_{\uparrow\cdots\uparrow 0 \downarrow \cdots \downarrow}(w_1, \dots, w_{2n}) &= \prod_{j=1}^n \left[\frac{qw_{n+1}}{w_j} \right] \prod_{j=n+2}^{2n+1} \left[\frac{qw_j}{w_{n+1}} \right] \\ &\times \tilde{\Xi}(w_1, \dots, w_n, w_{n+1}, w_{n+2}, \dots, w_{2n+1}).\end{aligned}$$

Here $\tilde{\Xi}(w_1, \dots, w_{2n+1})$ is a centred Laurent polynomial which can be easily determined from the recurrence properties of the renormalised eigenvector. Indeed, observe that the prefactor of $\tilde{\Xi}(w_1, \dots, w_{2n+1})$ in this equation is a centred Laurent polynomial in w_{n+1} of degree width $4n = 2(N-1)$, and hence saturates the degree width according to proposition

5.2. Therefore $\tilde{\Xi}(w_1, \dots, w_{2n+1})$ cannot depend on w_{n+1} , and may be evaluated by sending w_{n+1} to infinity or zero. In these limits, we know the left-hand side from proposition 5.3, and conclude that the unknown function $\tilde{\Xi}$ is given by the simple component (36) for even size $N - 1 = 2n$ with slightly rearranged arguments. After a short calculation we find our final result for $N = 2n + 1$:

$$\begin{aligned} \tilde{\Psi}_{\uparrow \dots \uparrow 0 \downarrow \dots \downarrow}(w_1, \dots, w_{2n+1}) &= \prod_{j=1}^n \left[\frac{qw_{n+1}}{w_j} \right] \prod_{j=n+2}^{2n+1} \left[\frac{qw_j}{w_{n+1}} \right] \\ &\times \tilde{\Psi}_{\uparrow \dots \uparrow \downarrow \dots \downarrow}(w_1, \dots, w_n, w_{n+2}, \dots, w_{2n+1}). \end{aligned} \quad (37)$$

5.4 The homogeneous point

In this section, we consider the homogeneous point $w_1 = \dots = w_N = 1$. In this case, the vector $|\tilde{\Psi}(1, \dots, 1)\rangle$ is an eigenstate of the spin-chain Hamiltonian with eigenvalue zero in the pseudo-momentum sector where $S' \equiv (-1)^{N+1}$, and thus a supersymmetry singlet. Our aim is to obtain the expressions for the simple components (5) and the sum rule (6). To this end, we need to control the normalisation of the vector in the homogeneous case. We show here that there is an redundant overall factor for all components when all w 's are equal. The case of $N = 3$ sites provides a simple illustration :

$$\tilde{\Psi}_{\uparrow 0 \downarrow}(1, 1, 1) = \tilde{\Psi}_{\downarrow 0 \uparrow}(1, 1, 1) = [q]^3 \times 1, \quad \tilde{\Psi}_{000}(1, 1, 1) = [q]^3 \times x, \quad x = q + q^{-1}.$$

If we remove the factor $[q]^3$ then we recover the components of the eigenvector of the Hamiltonian $|\Phi(x)\rangle$ for three sites from section 2. This can be done systematically for all N as we show hereafter. After this, we discuss the results obtained in sections 5.2 and 5.3, and relate the supersymmetry singlet and enumeration problems of alternating sign matrices.

Limit of the renormalised Bethe vector. From its definition we find that the homogeneous limit of the renormalised Bethe vector is given by

$$|\tilde{\Psi}(1, \dots, 1)\rangle = [q]^{N(N-1)/2} x^{-N/2} \beta(x)^N |\wedge\rangle, \quad x = q + q^{-1}, \quad (38)$$

where $\beta(x) = \mathcal{B}(1|1, \dots, 1)/[q]^N$. More explicitly, we have

$$\beta(x) = {}_a\langle \uparrow | \rho_{a,N}(x) \cdots \rho_{a,1}(x) | \downarrow \rangle_a, \quad \rho_{a,j}(x) = [q]^{-1} R_{a,j}^{(1,2)}(q^{-1}). \quad (39a)$$

In order to show that this depends only on x it is sufficient to use the definition (8): we find

$$\rho(x) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x^{1/2} & 0 & 0 \\ 0 & 0 & -1 & 0 & x^{1/2} & 0 \\ 0 & x^{1/2} & 0 & -1 & 0 & 0 \\ 0 & 0 & x^{1/2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (39b)$$

It is easily seen that when acting on a state $|\epsilon\rangle \otimes |\sigma\rangle$ with $\epsilon = \uparrow, \downarrow$ and $\sigma = \uparrow, 0, \downarrow$ that a spin flip of any type is weighted by $x^{1/2}$. For example $\rho(x)(|\uparrow\rangle \otimes |\downarrow\rangle) = -|\uparrow\rangle \otimes |\downarrow\rangle + x^{1/2} |\downarrow\rangle \otimes |0\rangle$. The way the operator $\beta(x)$ flips spins is rather non-local. Yet it is quite straightforward to determine certain properties of its action on simple basis vectors of V :

Lemma 5.6 *Let $|\sigma_1, \dots, \sigma_N\rangle$ be a basis vector of V , then the state $\beta(x)|\sigma_1, \dots, \sigma_N\rangle$ is $x^{1/2}$ times a polynomial in x with integer coefficients.*

Proof: The action of the operator $\beta(x)$ onto the basis state leads to a superposition of states where the spin components are flipped. From its construction (39), it is evident that any spin flip is weighted by $x^{1/2}$. A given spin component σ_j may either be lowered if $\sigma_j = \uparrow, 0$ or raised in case of $\sigma_j = 0, \downarrow$. Let us now consider those resulting vectors where exactly $k = 0, 1, \dots$ components are raised. This comes with a weight $x^{k/2}$. Moreover, their positions divide the chain into $k + 1$ subsegments of length at least one whose horizontal ends have exactly the same boundary conditions as the operator \mathcal{B} or β . This can easily be seen graphically by employing (23): an example with $k = 2$ is shown in figure 5. As $\beta(x)$



Figure 5: Example for $k = 2$ raising spin flips and the subdivision into $k + 1 = 3$ segments.

decreases by construction the magnetisation of the state by one, we conclude that one these subsegments exactly $k + 1$ spin components need to be lowered. These spin flips come with a weight $x^{(k+1)/2}$. Therefore we obtain the total weight $x^{k/2} \times x^{(k+1)/2} = x^{1/2} \times x^k$ where k is an integer. Taking into account all possible configurations, we conclude that the overall result is necessarily $x^{1/2}$ times a polynomial. The statement that it has integer coefficients follows from the fact that $\rho(x)$ is a linear function of $x^{1/2}$ whose coefficients are matrices with integer entries. \square

Next, we apply the operator $\beta(x)$ exactly N times to the reference state. Using the preceding lemma we obtain immediately the following:

Proposition 5.7 *The supersymmetry singlet*

$$|\Phi(x)\rangle = x^{-N/2} \beta(x)^N |\wedge\rangle = [q]^{-N(N-1)/2} |\tilde{\Psi}(1, \dots, 1)\rangle \quad (40)$$

is a polynomial in $x = q + q^{-1}$ with integer coefficients.

Simple components and square norm. We would like to evaluate the square norm $Z(w_1, \dots, w_N)$ and the components $\tilde{\Psi}_{\uparrow \dots \uparrow \downarrow \dots \downarrow}(w_1, \dots, w_{2n})$ or $\tilde{\Psi}_{\uparrow \dots \uparrow 0 \downarrow \dots \downarrow}(w_1, \dots, w_{2n+1})$ in the case $w_1 = \dots = w_N = 1$. From our results in the preceding sections we see that this amounts to analyse the partition function of the six-vertex model with domain wall boundary conditions in the homogeneous limit. Its relation to weighted enumeration of alternating sign matrices is well known (see e.g. [32]).

The relation is established as follows. In the homogeneous case the vertex weights become $\mathbf{a}(1) = \mathbf{b}(1) = [q]$ for the first four vertex configurations, and $\mathbf{c}(1) = [q^2]$ for the fifth and sixth vertex configuration shown in figure 6 here below. Let us evaluate the weight of a single configuration on an $N \times N$ square with say k vertices of type six. It is not difficult to see that the boundary conditions imply that there are thus $N + k$ vertices of type five, and $N^2 - N - 2k$ vertices of the other types. We conclude that the weight of the configuration is therefore

$$[q]^{N^2 - N - 2k} [q^2]^{N+k} [q^2]^k = [q]^{N(N-1)} [q^2]^N (x^2)^k, \quad x = q + q^{-1}.$$

It is known that any such configuration is in one-to-one correspondence with an $N \times N$ alternating sign matrix containing exactly k entries -1 [39]. The correspondence between vertex configurations and matrix entries is shown in figure 6, and we give an example in figure 4. Therefore, in the homogeneous limit the Izergin-Korepin determinant reduces to

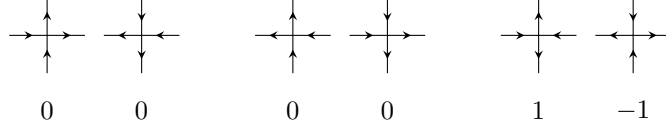


Figure 6: Correspondence between vertex configurations and the entries of alternating sign matrices.

a sum over all $N \times N$ alternating sign matrices, each of which is weighted by the weight $[q]^{N(N-1)}[q^2]^N(x^2)^k$ where k is the number of minus signs. This leads to the type of weighted enumeration which we discussed in section 2. In terms of the generating function $A_N(t)$ one finds therefore the partition function

$$Z_{\text{IK}}(w_1 = 1, \dots, w_N = 1) = [q]^{N(N-1)}[q^2]^N A_N(t = x^2).$$

This relation can now be applied to the vector $|\Phi(x)\rangle$ defined in (40): sending all inhomogeneity parameters to one in (36) and (37) we find

$$\Phi_{\uparrow \dots \uparrow \downarrow \dots \downarrow}(x) = \Phi_{\uparrow \dots \uparrow \downarrow \dots \downarrow}(x) = A_n(x^2),$$

which is precisely (5). In fact, the constant term of these polynomials equals $n!$, because any alternating sign matrix without minus signs is necessarily a permutation matrix. Moreover, the degree of the polynomial is $\lfloor (n-1)^2/4 \rfloor$. Hence our normalisation criteria for the components from section 2 are fulfilled, and the state $|\Phi(x)\rangle$ is indeed a polynomial in x with integer coefficients as established in proposition 5.7. Eventually, taking the homogeneous limit in proposition 5.5, we obtain

$$\|\Phi(x)\|^2 = A_N(x^2)$$

which is the sum rule stated in (6).

6 Conclusion

In this article we showed that for a particular twisted boundary condition the transfer matrix of the inhomogeneous nineteen-vertex model possesses a simple eigenvalue with a quite non-trivial corresponding eigenvector. At the homogeneous point, it is a supersymmetry singlet of the spin-one XXZ Hamiltonian, and its square norm as well as a particular component are given by generating functions for a type of weighted enumeration of alternating sign matrices, the weight being related to the anisotropy of the spin chain.

There are various open problems and generalisations. First of all, it might be interesting to characterise explicitly all components of the supersymmetry singlet in terms of combinatorial quantities, as was done for the spin-1/2 XXZ Hamiltonian with $\Delta = -1/2$ where a relation to certain refined enumerations of alternating sign matrices was found [40, 41]. For instance, in [9] certain components of the singlet were conjectured to be given by generating functions

for weighted enumeration of vertically-symmetric alternating sign matrices. Second, one may ask if there are other boundary conditions/twists which allow for simple eigenvalues of the transfer matrix. The answer is affirmative: in fact, one may study systematically the boundary conditions which are compatible with dynamic lattice supersymmetry and admit supersymmetry singlets [42], and use them in order to determine boundary conditions for which the transfer matrices of the corresponding inhomogeneous vertex model are expected to have a simple eigenvalue. Indeed, the nineteen-vertex model admits a non-diagonal twist which has all these features. This will be discussed in a forthcoming paper [43]. Third, the lattice supersymmetry of the spin-one XXZ chain survives the deformation to the quantum integrable spin-one XYZ chain [9]. We expect the transfer matrix of the corresponding vertex model to have a simple eigenvalue but very non-trivial eigenvector, too, similarly to the case of the eight-vertex model along a particular line of couplings [18]. Eventually, it would be interesting to understand the role which dynamic lattice supersymmetry plays in all this. While this type of supersymmetry is present in quite a few integrable spin chains, its interplay with integrability is not very well understood (see for instance the discussion in [25]), not to mention the question if there is a general underlying structure in the case of inhomogeneous models which reduces to the lattice supersymmetry when approaching the homogeneous point.

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